Calculus Application Problem - Don't Catch It!

1. The Problem: Imagine that you are one of many people at a party and that, unknown to everyone else, one person arrives carrying an infectious disease. How quickly will the disease spread, and what are the chances that you will leave the party with the disease?

2. Description of the Discrete Model:
The goal is to create a mathematical model that describes the spread of a disease at a party and also provides a crude description of real diseases. Suppose there are N people at the party and that the party is divided into M "stages". Let A_n represent the number of infected people at stage n, where $n=0,1,2,\ldots,M$. Let the initial condition $A_0=1$, which means that at the start of the party ($n=0$) there is one infected person. The task is to devise a rule that tells us the number of infected people at stage n+1 if we know the number of infected people at stage n. A rule that can be used to model this scenario is:

$\mathbf{A_{n+1} = A_n + kA_n(N - A_n)}$, for $n=0,1,2,\ldots,M-1$, and where **k** is a proportionality constant, such that $0<k<1$.

This relationship that describes how the number of infected people evolves is called a <u>difference</u> or <u>discrete equation</u> because it involves no derivatives and determines the number of infected persons at discrete time steps.

3. An Example of the Discrete Model:
To see how this difference equation works, on another sheet of paper create a chart with headings n (Stage Number), and A_n (Number of infected persons at stage n). Let's assume that N (the number of people at the party) is 50, let k (the proportionality constant) be 0.021, and let $A_0=1$. The first row of your chart should have $n=0$ and $A_0=1$. Using the difference equation above, fill in the table to show the spread of the disease until the entire population has been infected. Note: While filling in the table, round to the nearest whole number since decimals do not make sense in this problem. On your calculator, make a scatterplot of the values in your table.

4. Description of the Continuous Model:
The previous model, which involved a difference equation, is called discrete because it gives the infected population at distinct (discrete) stages in time. The solution seems to "jump" at each stage. Another model would have the infected population increase "smoothly", or continuously. To obtain this mathematical model, we can use a <u>differential equation</u>. An argument, very similar to the one that would be used to obtain the discrete model, can be used to derive the differential equation that describes the growth of the infected population. We now let A(t) denote the number of infected people at time $t\geq 0$. The corresponding differential equation is:

$\mathbf{A'(t)=kA(N-A)}$, where the initial condition is $A(0)=1$, and the coefficient $k\geq 0$.

Note: The constant **k** in the continuous model is not the same constant as the proportionality constant in the discrete model.

5. Solving the Differential Equation:
On another sheet of paper, by hand (without technology) solve the differential equation. (You may want to write **A'(t)** as **dA/dt**.) Show that the solution to the differential equation can be written as:

$$A(t) = \frac{N}{1+Be^{-k\cdot N\cdot t}}$$

Be sure to show all steps in your analysis.

6. Data Collection Chart: **Your ID Number:________**

Let N= the number of students in class, and let A(0)=1.

Data # 1

Stage Number	Number of Newly Infected Individuals	Number of Total Infected Individuals
0		
1		
2		
3		
4		
5		
6		

Let N=60, and let A(0)=3.

Data # 2

Stage Number	Number of Newly Infected Individuals	Number of Total Infected Individuals
0		
1		
2		
3		
4		
5		
6		
7		
8		
9		

7. Analyzing the Data:

Use the data from the first chart.

1. Make a scatterplot of the "Stage Number" vs. the "Number of Total Infected Individuals".

2. Since the data (should) appear to be a model for a logistic function, we need to find a function in the form:

$$A(t) = \frac{c}{1 + a \cdot e^{-b \cdot t}}$$

Therefore, we need to find values for the three constants a, b, and c. The value of c should be easy. For our activity,

c= ________

To find a and b, select two other points from the table/scatterplot. (One should be the initial point, and the other should (possibly) be the next to the last point.) Substitute the ordered pairs into the equation and solve for a and b. Round your answers to two decimal places. Show your work on paper.

a= ________ b= ________ The final equation is: A(t) = ______________________________

Of course, graph it to see how it fits the scatterplot.

3. Find a logistic regression equation on your calculator, and see if it fits the data.

The regression equation is: A(t) = ______________________________

4. As another check, substitute the value of **N** (called **c** in your solution) and **k** (you need to find this from your solution above) into the differential equation **A'(t)=kA(N-A)**, and plot the **slopefield** of the differential equation on your calculator. Also plot your solution with the slopefield.

5. Using the data from the second chart, make a scatterplot of this data and repeat problem #2 above to find the logistic function that models this new data. Be sure to show your work neatly on paper.

6. How do the two models compare? How are they the same and how are they different?